AMPLITUDE–PHASE RELATIONS IN THE ACOUSTIC FIELDS OF ULTRASONIC MEASURING SYSTEMS

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The acoustic field of an ultrasonic measuring system having symmetrical transmitting and receiving devices is analyzed. The fundamental field equation is derived, along with relations for the diffraction-induced phase changes and amplitude of the information-bearing signal.

For acoustic field investigations we adopt a symmetrical system [1] having identical piston-type transmitting and receiving devices, which are separated by a distance x greater than their diameter and much greater than the acoustic wavelength λ.

The determination of the parameters of an acoustic field requires that the wave equation for an infinite elastic medium be solved:

$$\Delta\psi + k^2\psi = 0, \tag{1}$$

in which ψ is the particle-velocity potential and k is the wave number, which is equal to $2\pi/\lambda$.

To simplify the analysis we assume that the investigated medium is perfectly elastic, homogeneous, and isotropic. The attenuation α of ultrasonic waves in the medium can be taken into account by the addition of an imaginary component $i\alpha$ to the wave number k.

The boundary conditions state a uniform distribution of the particle-velocity amplitude $U_0\exp(i\omega t)$ over the frontal surface of the transmitter as it executes harmonic oscillations at a cyclic frequency ω, and the absence of those oscillations outside the transmitting disk of radius a:

$$-\left.\frac{\partial\psi}{\partial x}\right|_{x=0} = \begin{cases} U_0\exp(i\omega t) & \text{for} \quad 0 \leqslant r < a, \\ 0 & \text{for} \quad r > a. \end{cases}$$

By the separation of variables and representation of the particle-velocity potential in a cylindrical coordinate system in the form $\psi = \psi_1(r)\psi_2(x)$ we transform the wave equation (1) into the following set of equations:

$$\begin{gathered} \frac{d^2\psi_1}{dr^2} + \frac{1}{r}\frac{d\psi_1}{dr} + \eta^2\psi_1 = 0, \\ \frac{d^2\psi_2}{dx^2} + (k^2 - \eta^2)\psi_2 = 0, \end{gathered} \tag{2}$$

in which η is a parameter independent of the coordinates.

These equations have the general solutions

$$\psi_1(r) = \varepsilon_1 J_0(\eta r) + \varepsilon_1' N_0(\eta r),$$

$$\psi_2(x) = \varepsilon_2\exp(-x\sqrt{\eta^2 - k^2}) + \varepsilon_2'\exp(x\sqrt{\eta^2 - k^2}),$$

where ε_1, ε_1', ε_2, and ε_2' are arbitrary constants.

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Since the zero-order Neumann function increases without bound as $r \to 0$, while the particle-velocity potential must maintain a finite value on the transmitter axis, we must set ε_1' equal to zero. At an infinite distance the wave is completely extinct, and $\exp(x\sqrt{\eta^2-k^2}) \to \infty$, so that $\varepsilon_2' = 0$ as well.

Consequently, the particular solutions of the wave equation represent the products of zero-order Bessel functions $\varepsilon_1 J_0(\eta r)$ and exponentials $\varepsilon_2 \exp(-\sqrt{\eta^2-k^2})$, in which the independent parameter η is indeterminate. For radiation into a half-space this parameter varies continuously, assumes any values, and determines the particular solutions. The sum of infinitely many of the latter, comprising the general solution of the wave equation, is written in integral form:

$$\psi(r, x) = \int_0^\infty \varepsilon_1\varepsilon_2 J_0(\eta r) \exp(-x\sqrt{\eta^2-k^2})\, d\eta. \tag{3}$$

In the expression obtained for the particle-velocity potential the product $\varepsilon_1\varepsilon_2$ of arbitrary constants, given the boundary condition

$$-\left.\frac{\partial\psi}{\partial x}\right|_{x=0} = \int_0^\infty \frac{\varepsilon_1\varepsilon_2\sqrt{\eta^2-k^2}}{\eta} J_0(\eta r)\,\eta d\eta = U_0 \exp(i\omega t) \qquad \text{for} \quad 0 \leqslant r < a$$

can be evaluated according to the Hankel inversion equation:

$$\varepsilon_1\varepsilon_2 = \frac{\eta}{\sqrt{\eta^2-k^2}} \int_0^a U_0 \exp(i\omega t) J_0(\eta r)\, r dr = \frac{aU_0 \exp(i\omega t)}{\sqrt{\eta^2-k^2}} J_1(\eta a). \tag{4}$$

The parameters of the information-bearing (useful) signal from the ultrasonic measuring system are determined by the amplitude and phase of the average acoustic pressure on the receiver, $\bar{p}$, which is expressed in terms of the acoustic pressure $p(r, x)$ at separate points of the axisymmetrical acoustic field in a plane orthogonal to the transmitter axis:

$$\bar{p} = \frac{2}{a^2} \int_0^a p(r, x)\, r dr. \tag{5}$$

From expressions (3)-(5), using the relation between the acoustic pressure and particle-velocity potential, $p(r, x) = ik(p_0/U_0)\psi(r, x)$, we determine the functional dependence of the average receiver pressure on the coordinate x and parameter η:

$$\bar{p} = 2ikap_0 \int_0^\infty \frac{\exp(i\omega t - x\sqrt{\eta^2-k^2})}{\sqrt{\eta^2-k^2}} \frac{J_1^2(\eta a)}{\eta a}\, d\eta, \tag{6}$$

where p_0 is the acoustic pressure amplitude on the frontal surface of the transmitter.

Reducing the power of the first-order Bessel function in expression (6) by the transformation [2]

$$\frac{J_1^2(\eta a)}{\eta a} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin\beta \cos^2\beta\, J_1(2\eta a \sin\beta)\, d\beta$$

with the introduction of the variable of integration β, we arrive at the result

$$\bar{p} = \frac{4ikp_0}{\pi} \exp(i\omega t) \int_0^{\frac{\pi}{2}} \cos^2\beta\, d\beta \int_0^\infty b\, \frac{\exp(-x\sqrt{\eta^2-k^2})}{\sqrt{\eta^2-k^2}} J_1(b\eta)\, d\eta, \tag{7}$$

in which $b = 2a\sin\beta$.

The improper integral in Eq. (7) is evaluated by multiplying the left- and right-hand sides of the following well-known type of relation in the theory of Bessel functions by rdr:

$$\int_0^\infty \frac{\exp(-x\sqrt{\eta^2-k^2})}{\sqrt{\eta^2-k^2}} J_0(\eta r)\,\eta d\eta = \frac{\exp(-ik\sqrt{x^2+r^2})}{\sqrt{x^2+r^2}}$$

and integrating them from 0 to b:

$$\int_0^\infty b \frac{\exp(-x\sqrt{\eta^2-k^2})}{\sqrt{\eta^2-k^2}} J_1(b\eta)\, d\eta = \frac{1}{ik} [\exp(-ikx) - \exp(-ik\sqrt{x^2+b^2})]. \tag{8}$$

From expression (7) and (8), substituting the variable of integration $\beta = \theta/2$, we deduce the initial formula for the average acoustic pressure:

$$\bar{p} = p_0 \exp[i(\omega t - kx)] - \frac{p_0}{\pi} \exp(i\omega t) \int_0^\pi (1+\cos\theta) \exp(-ik\sqrt{x_1^2 - 2a^2\cos\theta})\, d\theta, \tag{9}$$

in which $x_1 = \sqrt{x^2 + 2a^2}$.

This formula indicates that the average pressure on the receiver has two components, the first of which corresponds to a plane wave having an infinite front. The second component corresponds to diffracted waves, and its value p_d determines the diffraction variations of the amplitude and phase of the useful signal.

Replacing the radical in the exponent by a power series with four terms retained in the expansion, we transform the diffraction component as follows:

$$p_d = -\frac{p_0}{\pi} \exp[i(\omega t - kx - q)] \int_0^\pi \exp[i(q\cos\theta - \gamma\sin^2\theta)](1+\cos\theta)\, d\theta, \tag{10}$$

where

$$q = \frac{ka^2}{x_1}\left(1 + \frac{a^4}{2x_1^4}\right) \quad \text{and} \quad \gamma = \frac{ka^4}{2x_1^3}\left(1 + \frac{a^2\cos\theta}{2x_1^2}\right).$$

Then, replacing $\exp(-i\gamma\sin^2\theta)$ by a power series with l terms of the expansion and executing appropriate transformation, we find in place of the integral on the right-hand side of Eq. (10) a set of $2l + 6$ integrals of two types, expressed directly in terms of gamma functions and Bessel functions:

$$\int_0^\pi \exp(iq\cos\theta)\sin^{2m}\theta\, d\theta = \Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\left(\frac{2}{q}\right)^m J_m(q),$$

$$\int_0^\pi \exp(iq\cos\theta)\sin^{2m}\theta\cos\theta\, d\theta = i\pi \frac{2^{m+1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}{(2m+1)q^m} J_{m+1}(q)$$

with indices m = 0, 1, 2, . . . , l + 2. It is verified by analysis that when the inequalities

$$\frac{2a}{x} \leqslant 1 \quad \text{and} \quad \frac{ka^4}{2x_1^3} \leqslant \pi$$

hold, the values of the integrals with nonzero indices can be neglected on account of their smallness, and

$$p_д = -p_0 \exp[i(\omega t - kx - q)][J_0(q) + iJ_1(q)]. \tag{11}$$

The fundamental equation for the acoustic field on an ultrasonic measuring system with allowance for attenuation can therefore be represented in the form

$$\bar{p} = p_0 \exp\left[i\left(\omega t - \frac{2\pi x}{\lambda}\right) - \alpha x\right]\{1 - [J_0(q) + iJ_1(q)]\exp(-iq)\}$$

or

$$\bar{p} = \varepsilon p_0 \exp\left[i\left(\omega t - \frac{2\pi x}{\lambda} + \varphi'\right) - \alpha x\right], \tag{12}$$

where

$$\varepsilon = \sqrt{1 + J_0^2(q) + J_1^2(q) - 2\,[J_0(q)\cos q + J_1(q)\sin q]}\ ; \tag{13}$$

$$\varphi' = \operatorname{arctg} \frac{J_0(q)\sin q - J_1(q)\cos q}{1 + J_1(q)\sin q - J_0(q)\cos q}\ . \tag{14}$$

These relations make it possible to determine precisely the diffraction attenuation ε of the amplitude and the phase shift φ' of the useful signal from an ultrasonic measuring system. The results of calculations of these parameters exhibit good agreement with experiment.

LITERATURE CITED

1. N. I. Brazhnikov, Ultrasonic Techniques [in Russian], Énergiya, Moscow (1965).
2. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], GIFML (1963).